THE EFFECT OF ROTATING DISTANT MASSES IN EINSTEIN'S THEORY OF GRAVITATION

by

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The reason for this paper may best be illustrated by a quote from Einstein's fundamental paper of 1914.[34] By way of introduction, he writes:

> In the first place it seems that such an extension of the theory of relativity should be turned down on physical grounds. Let K be a Galilean-Newtonian coordinate system, and let K' be a coordinate system rotating uniformly relative to K. Then centrifugal forces would be in effect for masses at rest in the K' coordinate system, while no such forces would be present for objects at rest in K. Already Newton viewed this as proof that the rotation of K' had to be considered as "absolute," and that K' could not then be treated as the "resting" frame, K. Yet as E. Mach has shown, this argument is not sound. One need not view the existence of such centrifugal forces as originating from the motion of k"; one could just as well account for them as resulting from the average rotational effect of distant, detectable masses as evidenced in the vicinity of K', whereby K' is treated as being at rest. If Newtonian mechanics disallow such a view, then this could very well be the foundation for the defects of that theory....

Einstein's theory appears to have been completely developed in the 1915 publications, and so the obvious question is: is the new theory really so free from the shortcomings of the Newtonian one that according to its equations the rotation of distant masses indeed does generate a gravitational field that equals the centrifugal field? One is probably tempted to consider a discussion of this question as vain labor since the required equivalence appears to be guaranteed by the general covariance of the field equations. The matter is not that clear-cut, however, because the boundary conditions for $g_{\mu\varpi}$ become important if space is taken to be infinite. The fundamental questions which have a bearing upon this have been considered by deSitter[35] and Einstein[36] so that we shall not concern ourselves with these general equations. We will, rather, be concerned with the mathematical development of a rotational field of distant masses for a specific example. To this end Einstein's method of integrating the field equations serves admirably.[37] For our case we choose the field inside a uniformly rotating, infinitesimally thin, hollow sphere of uniform surface density.

In the first section of this paper (which may be skipped without loss of comprehension in the subsequent parts) the mean value theorem is applied to $g_{\mu\nu}$ for the inside of the spherical shell. The second part discusses the movement of a point mass in said field.

A. Theoretical Development: The Calculation of $g_{\mu\nu}$ for the Vicinity of the Central Point of a Rotating Hollow Sphere

Symbol key:

a	radius of the hollow sphere
M	its mass
ω	its angular velocity
x, y, z	Cartesian coordinates of an orbiting point on the surface of the sphere
x_0, y_0, z_0	coordinates of an interior point
$\varkappa$	gravitational constant
ρ_0	mean interior density.

In addition, mention must be made of assumptions and approximations used in computing the field strength. Near the center of the sphere the field is assumed to be so weak that we need only concern ourselves to first order in $\gamma_{\mu\nu}$ (where $\gamma_{\mu\nu}$ is defined by $g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu}$). This allows Einstein's approximation method to be applied in integrating the field equations. Secondly, it is assumed that the velocity components of the affected masses are small relative to the speed of light. In the limit, this would yield the Newtonian result in which the first order potentials may be ignored. This second assumption, which is totally independent of the first, will only be used to cancel third and higher orders in v/c. Finally: the calculations relate to the vicinity of the center of the sphere. Let r be the distance between the point (x_0, y_0, z_0) and the sphere's center, and let R be the distance from said point to the element of integration; then we shall develop 1/R as a power series in r/a which we shall terminate after the second order.

Einstein's method of integration by approximation provides us with the following conditions on $g_{\mu\nu}$:

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu}, \quad \begin{aligned} \delta_{\mu\nu} &= 1, \ \mu = \nu, \\ &= 0, \ \mu \neq \nu, \end{aligned} \tag{1}$$

$$\gamma_{\mu\nu} = \gamma_{\mu\nu}' - \frac{\delta_{\mu\nu}}{2} \sum_{\alpha} \gamma_{\alpha\alpha}', \tag{2}$$

$$\gamma_{\mu\nu}' = -\frac{\varkappa}{2\pi} \int \frac{T_{\mu\nu}(x, y, z, t - r)}{R} dV_0. \tag{3}$$

where $T_{\mu\nu}$ is the covariant mass energy tensor, dV_0 us the normal spatial volume element in the integration space (in polar coordinates this is $r^2 dr \sin\theta d\theta d\phi$); and

$$R^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2.$$

The coefficients $g_{\mu\nu}$ of the line element are related to the coordinates $x_1=x$, $x_2=y$, $x_3=z$ and $x_4=it$.

In accordance with the first assumption one may substitute the contra-variant energy tensor for the co-variant one, so that the former, ignoring the potentials, is given by:

$$\left. \begin{aligned} T_{\mu\nu} = T^{\mu\nu} = \varrho_0 \frac{dx_\mu}{ds} \frac{dx_\nu}{ds} &= \\ = \varrho_0 \frac{dx_\mu}{dx_4} \frac{dx_\nu}{dx_4} \left(\frac{dx_4}{ds}\right)^2 &. \end{aligned} \right\}$$

If the hollow sphere rotates around the z-axis with angular velocity ω, then for one of its points with polar coordinates α, θ, ϕ:

$$\left. \begin{aligned} \frac{dx_1}{dx_4} &= -i \frac{dx}{dt} = i a \omega \sin\vartheta \sin\varphi, \\ \frac{dx_2}{dx_4} &= -i \frac{dy}{dt} = -i a \omega \sin\vartheta \cos\varphi, \\ \frac{dx_3}{dx_4} &= 0. \end{aligned} \right\} \tag{5}$$

which, when substituted into 4), solves for $T_{\mu\nu}$ as follows:

$$T_{\mu\nu} = \varrho_0 \left(\frac{dx_4}{ds}\right)^2 \begin{Bmatrix} -a^2\omega^2 \sin^2\vartheta \sin^2\varphi, & +a^2\omega^2 \sin^2\vartheta \sin\varphi\cos\varphi, & 0 & i a\omega \sin\vartheta \sin\varphi \\ +a^2\omega^2 \sin^2\vartheta \sin\varphi\cos\varphi, & -a^2\omega^2 \sin^2\vartheta \cos^2\varphi, & 0 & -i a\omega \sin\vartheta \cos\varphi \\ 0 & 0 & 0 & 0 \\ i a\omega \sin\vartheta \sin\varphi, & -i a\omega \sin\vartheta \cos\varphi, & 0 & 1 \end{Bmatrix}. \tag{6}$$

Since ρ_0 is the effective density of the sphere, in order to maintain the tensor characteristics of the integral in 3), we must substitute the effective volume element for dV_0. For this the appropriate relation is (5):

$$dV_0 = \sqrt{g}\, i \frac{dx_4}{ds} dV. \tag{7}$$

To integrate polar coordinates are used so that:

$$dV\sqrt{g} = a^2 \, da \sin\theta \, d\theta \, d\phi. \tag{8}$$

Lastly, 1/R must still be expressed in terms of the integration variables. We choose the coordinate system in such a way that the reference point (x_0, y_0, z_0) falls in the Z-X plane, so that the coordinates are:

$$x_0 = r \sin\theta_0, \ y_0 = 0, \ z_0 = r \cos q_0.$$

Then

$$R^2 = (a \sin\vartheta \cos\vartheta - r \sin\vartheta_0)^2 + a^2 \sin^2\vartheta \sin^2\varphi + (a \cos\vartheta - r \cos\vartheta_0)^2 = a^2\left[1 - \frac{2r}{a}(\sin\vartheta \cos\varphi \sin\vartheta_0 + \cos\vartheta \cos\vartheta_0) + \frac{r^2}{a^2}\right].$$

Neglecting higher order terms the binomial expansion yields:

$$\frac{1}{R} = \frac{1}{a}\left\{1 + \frac{r}{a}(\sin\vartheta \cos\varphi \sin\vartheta_0 + \cos\vartheta \cos\vartheta_0) - \frac{1}{2}\frac{r^2}{a^2} + \frac{3}{2}\frac{r^2}{a^2}(\sin\vartheta \cos\varphi \sin\vartheta_0 + \cos\vartheta \cos\vartheta_0)^2\right\}. \quad (9)$$

Denoting the expression within the braces as K allows us to write:

$$1/R = K/a. \quad (9a)$$

Substituting 6), 7), 8) and 9a) into 3) yields:

$$\left.\begin{aligned}
\gamma_{11}' &= \frac{\varkappa}{2\pi}\varrho_0 a^3 \omega^2 da \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \left(\frac{dx_4}{ds}\right)^2 \sin^3\vartheta \sin^2\varphi\, K, \\
\gamma_{22}' &= \frac{\varkappa}{2\pi}\varrho_0 a^3 \omega^2 da \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \left(\frac{dx_4}{ds}\right)^2 \sin^3\vartheta \cos^2\varphi\, K, \\
\gamma_{44}' &= -\frac{\varkappa}{2\pi}\varrho_0 a\, da \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \left(\frac{dx_4}{ds}\right)^2 \sin\vartheta\, K, \\
\gamma_{12}' &= -\frac{\varkappa}{2\pi}\varrho_0 a^3 \omega^2 da \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \left(\frac{dx_4}{ds}\right)^2 \sin^3\vartheta \sin\varphi \cos\varphi\, K, \\
\gamma_{14}' &= \frac{\varkappa}{2\pi i}\varrho_0 a^2 \omega\, da \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \left(\frac{dx_4}{ds}\right)^2 \sin^2\vartheta \sin\varphi\, K, \\
\gamma_{24}' &= -\frac{\varkappa}{2\pi i}\varrho_0 a^2 \omega\, da \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \left(\frac{dx_4}{ds}\right)^2 \sin^2\vartheta \cos\varphi\, K, \\
\gamma_{12}' &= \gamma_{23}' = \gamma_{33}' = \gamma_{43}' = 0.
\end{aligned}\right\} \quad (10)$$

The absolute value of the term dx_4/ds differs from unity only in terms of order $\omega^2 a^2$. It also appears as a factor in the first order of $\gamma_{\mu\nu}$. Thus it follows trivially from the scale element:

$$\left.\begin{aligned}
ds^2 &= -dx_1^2 - dx_2^2 - dx_3^2 - dx_4^2, \\
\frac{ds^2}{dx_4^2} &= -1 - \frac{dx_1^2 + dx_2^2 + dx_3^2}{dx_4^2} = \\
&= -1 + \omega^2 a^2 \sin^2\vartheta, \\
\frac{ds}{dx_4} &= i\left(1 - \frac{\omega^2 a^2}{2}\sin^2\vartheta\right); \\
\left(\frac{dx_4}{ds}\right)^3 &= i\left(1 + \frac{3}{2}\omega^2 a^2 \sin^2\vartheta\right).
\end{aligned}\right\} \quad (11)$$

Since we are dealing only in terms of order no higher than ω^2a^2, we can set $(dx_4/ds)^3 = i$ for all those $\gamma^3{}_{\mu\nu}$ which already contain the ωa factor. To solve for this we make use of expression 11). We set $\rho_0 da = \sigma$, and 10) becomes:

$$\left.\begin{aligned}
\gamma_{11}' &= -\frac{\varkappa}{2\pi i}\sigma a^3\omega^2\int_0^{2\pi} d\varphi\int_0^{\pi} d\vartheta \sin^3\vartheta\sin^2\varphi\, K\left(\frac{dx_4}{ds}\right)^{-1},\\
\gamma_{22}' &= -\frac{\varkappa}{2\pi i}\sigma a^3\omega^2\int_0^{2\pi} d\varphi\int_0^{\pi} d\vartheta \sin^3\vartheta\cos^2\varphi\, K\left(\frac{dx_4}{ds}\right)^{-1},\\
\gamma_{44}' &- \frac{\varkappa}{2\pi i}\sigma a\int_0^{2\pi} d\varphi\int_0^{\pi} d\vartheta \sin\vartheta\, K\left(1+\frac{3}{2}\omega^2a^2\sin^2\vartheta\right)\left(\frac{dx_4}{ds}\right)^{-1},\\
\gamma_{12}' &= \frac{\varkappa}{2\pi i}\sigma a^3\omega^2\int_0^{2\pi} d\varphi\int_0^{\pi} d\vartheta \sin^3\vartheta\sin\varphi\cos\varphi\, K\left(\frac{dx_4}{ds}\right)^{-1},\\
\gamma_{14}' &= \frac{\varkappa}{2\pi}\sigma a^2\omega\int_0^{2\pi} d\varphi\int_0^{\pi} d\vartheta \sin^2\vartheta\sin\varphi\, K\left(\frac{dx_4}{ds}\right)^{-1},\\
\gamma_{24}' &= -\frac{\varkappa}{2\pi}\sigma a^2\omega\int_0^{2\pi} d\varphi\int_0^{\pi} d\vartheta \sin^2\vartheta\cos\varphi\, K\left(\frac{dx_4}{ds}\right)^{-1},\\
\gamma_{13}' &= \gamma_{23}' = \gamma_{33}' = \gamma_{34}' = 0.
\end{aligned}\right\} \quad (12)$$

Substituting 9) into these for K and utilizing the integrals we arrive at:

$$\int \varrho_0\, dV_0 = i\int \varrho_0 \frac{dx_4}{ds}\, dV = M$$

(or, via equation 11: $M = 4\pi\sigma a^2(1 + \omega^2a^2/3)$.)

$$\left.\begin{aligned}
\gamma_{11}' &= -\frac{\varkappa}{2\pi}\frac{M}{3a}a^2\omega^2\left(1-\frac{r^2}{5a^2}\right),\\
\gamma_{22}' &= -\frac{\varkappa}{2\pi}\frac{M}{3a}a^2\omega^2\left\{1-\frac{r^2}{5a^2}(1-3\sin^2\vartheta_0)\right\},\\
\gamma_{44}' &= \frac{\varkappa}{2\pi}\frac{M}{a}\left\{1+a^2\omega^2\left[1-\frac{r^2}{5a^2}\left(1-\frac{3}{2}\sin^2\vartheta_0\right)\right]\right\},\\
\gamma_{24}' &= -\frac{i\varkappa}{2\pi}\frac{M}{3a}\omega r\sin\vartheta_0,\\
\gamma_{12}' &= \gamma_{14}' = \gamma_{13}' = \gamma_{23}' = \gamma_{33}' = \gamma_{43}' = 0.
\end{aligned}\right\} \quad (13)$$

Thus we can obtain the values $\gamma_{\mu\nu}$ via 1) and 2), and so solve for $g_{\mu\nu}$. Converting from polar to rectangular coordinates and replacing Einstein's gravitational constant, x, by k = x/(8π) (where c=1), the result is:

$$
\begin{aligned}
g_{11} &= -1 - \frac{2kM}{a}\left\{1 + a^2\omega^2 - \frac{\omega^2}{10}(2z_0^2 + x_0^2)\right\},\\
g_{22} &= -1 - \frac{2kM}{a}\left\{1 + a^2\omega^2 - \frac{\omega^2}{10}(2z_0^2 - 3x_0^2)\right\},\\
g_{33} &= -1,\\
g_{44} &= -1 + \frac{2kM}{a}\left\{1 + \frac{5a^2\omega^2}{3} - \frac{\omega^2}{6}(2z_0^2 - x_0^2)\right\},\\
g_{24} &= -i\frac{4kM}{3a}\omega x_0,
\end{aligned}
\tag{14}
$$

where all other $g_{\mu\nu}$ vanish.

The transformation:

$$
\left.
\begin{aligned}
x_1' &= x_1 \cos\alpha + x_2 \sin\alpha,\\
x_2' &= -x_1 \sin\alpha + x_2 \cos\alpha,\\
x_3' &= x_3,\\
x_4' &= x_4.
\end{aligned}
\right\}
\tag{15}
$$

rids us of the special choice of our system of coordinates (for we had placed our reference point (x_0, y_0, z_0) in the Z-X plane). Then, via the formula for transforming a covariant tensor of second order:

$$g_{\sigma\tau}' = (\partial x_\mu/\partial x_\sigma')\,(\partial x_\nu/\partial x_\tau')\,g_{\mu\nu}$$

the coefficient matrix becomes:

$$
g_{\mu\nu} = \left\{
\begin{matrix}
-1 - \frac{2kM}{a}\left[1 + \frac{a^2\omega^2}{3} - \frac{2\omega^2}{15}(z^2 + x^2 - 2y^2)\right], & +\frac{2kM}{a}\frac{\omega^2}{5}xy, & 0, & +i\frac{4kM}{3a}\omega y\\
+\frac{2kM}{a}\frac{\omega^2}{5}xy, & -1 - \frac{2kM}{a}\left[1 + \frac{a^2\omega^2}{3} - \frac{2\omega^2}{15}(z^2 + y^2 - 2x^2)\right], & 0, & -i\frac{4kM}{3a}\omega x\\
0, & 0, & 0, & 0,\\
+i\frac{4kM}{3a}\omega y, & -i\frac{4kM}{3a}\omega x, & 0, & -1 + \frac{2kM}{a}\left[1 + a^2\omega^2 - \frac{2\omega^2}{15}(2z^2 - x^2 - y^2)\right]
\end{matrix}
\right\}
\tag{16}
$$

The subscript 0 has been dropped from the coordinates so that x, y, and z will from now on signify the coordinates of the observer or reference point.

B. Physical Section: The Motion of a Point Mass Inside the Rotating Hollow Sphere

We shall develop the equations for the motion of a point mass in the vicinity of the center of our rotating spherical shell. The field in this vicinity is characterized by the coefficients of the matrix $g_{\mu\nu}$ (Equation 16 of part A).

The law governing the motion for a point mass is given by Einstein as the condition:

$$\delta \int ds = 0,$$

or, upon expanding 6):[38]

$$\frac{d^2 x_\tau}{ds^2} = \Gamma^\tau_{\mu\nu}\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}\ ^{1)}. \qquad \tau = 1 \ldots . 4. \tag{17}$$

For the field components $\Gamma^\tau_{\mu\nu}$ there applies, according to the first assumption:

$$
\Gamma^\tau_{\mu\nu} = -\left\{\begin{matrix}\mu\nu\\ \tau\end{matrix}\right\} = \left[\begin{matrix}\mu\nu\\ \tau\end{matrix}\right] = \frac{1}{2}\left(\frac{\partial g_{\tau\nu}}{\partial x_\mu} + \frac{\partial g_{\mu\tau}}{\partial x_\nu} - \frac{\partial g_{\mu\nu}}{\partial x_\tau}\right).
\tag{18}
$$

We only want to consider motions of the point mass which are small relative to the speed of light. Thus we may neglect second order terms in velocity. This allows us to conceal all the right-hand terms of equations 17) in which the index 4 does not occur. Moreover, we may replace the differentials in s by those in t. Recalling that $dx_4/dt = i$, makes equations 17) into:

$$\frac{d^2 x_\tau}{dt^2} = 2i\left(\Gamma^\tau_{14}\frac{dx_1}{dt} + \Gamma^\tau_{24}\frac{dx_2}{dt} + \Gamma^\tau_{34}\frac{dx_3}{dt}\right) - \Gamma^\tau_{44}. \tag{19}$$

In what follows we will only need the components of $\Gamma^\tau_{\mu\nu}$ which involve at least one 4 in their indices. That leaves us with 16 terms which (though they are not tensor components) in our case will render themselves in accordance with an anti-symmetric tensor matrix of second order. Then, in a stationary field, the partial derivatives in x_4 collectively disappear and the quantities $\Gamma^{\tau}_{\mu\nu}$ may be written as follows:[39]

$$\left.\begin{array}{llll}
\Gamma^1_{14}=0 & \Gamma^1_{24}=\frac{1}{2}\left(\frac{\partial g_{14}}{\partial x_2}-\frac{\partial g_{24}}{\partial x_1}\right) & \Gamma^1_{34}=\frac{1}{2}\left(\frac{\partial g_{14}}{\partial x_3}-\frac{\partial g_{34}}{\partial x_1}\right) & \Gamma^1_{44}=-\frac{1}{2}\frac{\partial g_{44}}{\partial x_1} \\
\Gamma^2_{14}=\frac{1}{2}\left(\frac{\partial g_{24}}{\partial x_1}-\frac{\partial g_{14}}{\partial x_2}\right) & \Gamma^2_{24}=0 & \Gamma^2_{34}=\frac{1}{2}\left(\frac{\partial g_{24}}{\partial x_3}-\frac{\partial g_{34}}{\partial x_2}\right) & \Gamma^2_{44}=-\frac{1}{2}\frac{\partial g_{44}}{\partial x_2} \\
\Gamma^3_{14}=\frac{1}{2}\left(\frac{\partial g_{34}}{\partial x_1}-\frac{\partial g_{14}}{\partial x_3}\right) & \Gamma^3_{24}=\frac{1}{2}\left(\frac{\partial g_{34}}{\partial x_2}-\frac{\partial g_{24}}{\partial x_3}\right) & \Gamma^3_{34}=0 & \Gamma^3_{44}=-\frac{1}{2}\frac{\partial g_{44}}{\partial x_3} \\
\Gamma^4_{14}=\frac{1}{2}\frac{\partial g_{44}}{\partial x_1} & \Gamma^4_{24}=\frac{1}{2}\frac{\partial g_{44}}{\partial x_2} & \Gamma^4_{34}=\frac{1}{2}\frac{\partial g_{44}}{\partial x_3} & \Gamma^4_{44}=0
\end{array}\right\} \quad (20)$$

If we combine values of $g_{\mu\nu}$ from equation 16) we obtain the following result:

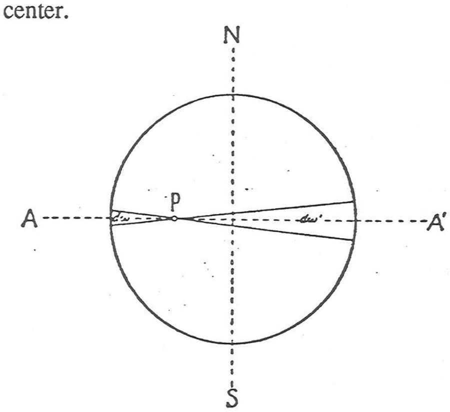
$$\left.\begin{array}{cccc}
0 & i\frac{4kM}{3a}\omega & 0 & -\frac{kM}{3a}\omega^2 x \\
-i\frac{4kM}{3a}\omega & 0 & 0 & -\frac{kM}{3a}\omega^2 y \\
0 & 0 & 0 & \frac{2kM}{3a}\omega^2 z \\
\frac{kM}{3a}\omega^2 x & \frac{kM}{3a}\omega^2 y & -\frac{2kM}{3a}\omega^2 z & 0
\end{array}\right\} \quad (21)$$

From equations 19) and 21) we now obtain the equation of motion for our point-mass as follows:

$$\left.\begin{array}{l}
\ddot{x} = -\frac{8kM}{3a}\omega\dot{y} + \frac{4kM}{15a}\omega^2 x \\
\ddot{y} = +\frac{8kM}{3a}\omega\dot{x} + \frac{4kM}{15a}\omega^2 y \\
\ddot{z} = \qquad\qquad -\frac{8kM}{15a}\omega^2 z
\end{array}\right\} \quad (22).$$

The right-hand terms of the equations represent the components of the force which the field exerts on the point of unit mass. As one can see, the first terms of the X and Y components correspond to the Coriolis force, and the second terms correspond to the centrifugal force. The third equation yields the surprising result that the centrifugal force possesses an axial component. Its appearance in the field of the rotating sphere may be explained as follows: as seen by an observer-at-rest, those surface elements of the hollow sphere which are nearest the equator have a greater velocity, and hence also a greater apparent (inertial and gravitational) mass than those about the poles. The field of a rotating hollow sphere of uniform surface density is therefore conformable to the field of a spherical shell at rest for which the surface density increases with increasing polar angle, θ. That is, points away from the equatorial plane are drawn towards the equatorial plane.

(We also note in passing that it is easy to visualize that in the interior of such a hollow sphere of unequal surface density, forces appear analogous to the centrifugal force.) It is well known that using the theory of potentials, we can show that in a hollow sphere, the gravitational force disappears, provided that the surface density is uniform. The force of attraction of the surface elements, which are within a solid angle, dω, is equal and opposite to the force exerted by the surface elements situated in the opposite angle, dω'. This is, of course, no longer the case for a nonuniform surface density. Let AA' be the equatorial plane, then, at P, there are surface elements in dω which, on the average, are closer to the equator and are thus of higher specific weight than those of dω'. Hence a net force results in the plane AA', that is to say, an outward force perpendicular to the axis of rotation, that becomes weaker the close the point P is to the center.



The fact that in nature we only have been able to observe a radial, but never an axial component of the centrifugal force can be brought into agreement with the results obtained here by noting that the approximation of the heaven of fixed stars by means of an infinitesimally thin hollow sphere is certainly not physical. But ever when we want to improve our approximation (possibly by means of a mass distribution,) the method of approximation used here will never obtain a field that is completely equivalent to a real centrifugal field. Such a field we can only obtain if we assume all the cosmic masses (Milky Way systems, etc.) are rotating and then to calculate their gravitational influences. The solution for the retarded potentials (equation 3) presupposes, however, that at infinity the boundary conditions $\gamma_{\mu\nu} = 0$ pertain. As Einstein has shown in his cosmological work,[40] these boundary conditions are approximately fulfilled for a coordinate system in which the fixed stars average at rest. Our solution 16) therefore does not represent the field of a hollow sphere "isolated in the cosmos," but the field inside such a hollow sphere outside which, at yet far greater distances, masses are present that are, on average, at rest relative to the chosen coordinate system.

The field represented by equation 16) is hence, by way of example, the one that would exist in the locality of the Sun's center if, instead of the Sun and all the planets, there would exist a great hollow sphere with a size of about Neptune's orbit, which sphere would, relative to the fixed stars, rotate with an angular velocity, ω. If in the center of this sphere observers would find themselves on a celestial body whose gravitational field could be ignored and which would be rotating around the same axis as the hollow sphere, then these observers would perceive centrifugal and Coriolis forces composed of the effects of their own rotation and those of the rotating hollow sphere. The influence of the field of the hollow sphere on the centrifugal field originating from the central body's self-rotation we shall examine below.

For that purpose we introduce a coordinate system which is firmly tied to the rotating body under consideration, which body rotates with angular velocity ω. This happens by means of the transformation:

$$\left.\begin{aligned} x' &= x\cos\omega'\frac{x_4}{i} + \sin\omega'\frac{x_4}{i}; \qquad z' = z, \\ y' &= -x\sin\omega'\frac{x_4}{i} + y\cos\omega'\frac{x_4}{i}; \quad x_4' = x_4. \end{aligned}\right\} \quad (23)$$

As a result of this transformation, the quantities $g_{\mu 4}$ which interest us are changed into:

$$\left.\begin{aligned} g_{14}' &= -iy'\left[\omega'\left(1+\frac{2kM}{3a}\right)-\omega\frac{4kM}{3a}\right], \\ g_{24}' &= ix'\left[\omega'\left(1+\frac{2kM}{3a}\right)-\omega\frac{4kM}{3a}\right], \\ g_{44}' &= -1+\frac{2kM}{a}\left[1+\frac{5a^2\omega^2}{3}-\frac{\omega^2}{3}z^2\right]+ \\ &\quad +(x'^2+y'^2)\left\{\omega'^2\left(1+\frac{2kM}{a}\right)- \right. \\ &\quad \left. -\omega\omega'\frac{4kM}{3a}+\omega^2\frac{kM}{3a}\right\}. \end{aligned}\right\} \quad (24)$$

In accordance with equations 19) and 20) we construe from these quantities the equations of motion:

$$\left.\begin{aligned}\ddot{x} &= 2\left[\omega'\left(1+\frac{2kM}{a}\right)-\omega\frac{4kM}{3a}\right]\dot{y}+\left\{\omega'^2\left(1+\frac{2kM}{a}\right)-\omega\omega'\frac{8kM}{3a}+\omega^2\frac{4kM}{15a}\right\}x\\ \ddot{y} &= -2\left[\omega'\left(1+\frac{2kM}{a}\right)-\omega\frac{4kM}{3a}\right]\dot{x}+\left\{\omega'^2\left(1+\frac{2kM}{a}\right)-\omega\omega'\frac{8kM}{3a}+\omega^2\frac{4kM}{15a}\right\}y\\ \ddot{z} &= -\frac{8kM}{3a}\omega^2 z.\end{aligned}\right\} \quad (25)$$

If here we take M=0, then we obtain the common centrifugal-Coriolis field:

$$\left.\begin{aligned}\ddot{x} &= 2\omega'\dot{y}+\omega'^2x,\\ \ddot{y} &= -2\omega'\dot{x}+\omega'^2y,\\ \ddot{z} &= 0.\end{aligned}\right\} \quad (26)$$

If we take $M\neq0$ and $\nu=0$, then we have:

$$\begin{aligned}\ddot{x} &= 2\omega'\left(1+\frac{2kM}{a}\right)\dot{y}+\omega'^2\left(1+\frac{2kM}{a}\right)x,\\ \ddot{y} &= -2\omega'\left(1+\frac{2kM}{a}\right)\dot{x}+\omega'^2\left(1+\frac{2kM}{a}\right)y,\\ \ddot{z} &= 0,\end{aligned} \quad (27)$$

which shows us how the inertial forces are influenced by the presence of the surrounding mass, M. The centrifugal force and Coriolis force are multiplied by the factor (1 + 2kM/a).

Finally, from equation 25) we can see that if body and sphere rotate in the same sense, then there results a reduction in the centrifugal and Coriolis forces. If we posit:

$$\omega' = \omega\frac{4kM}{3(2kM+a)}, \quad (28)$$

then the Coriolis force disappears. We could define 4kM/(3(2kM+a)) as the "drag coefficient" of the hollow sphere with respect to the Coriolis force. The centrifugal force cannot be made to disappear because the expressions within the braces of 25) have no real roots for ω if those expressions are set equal to zero. In a stationary frame of reference (ω' = 0), the value of the centrifugal force would be:

$$kM/(3a)\ \omega^2 \sqrt{(x^2 + y^2)}.$$

If we, in the same manner, let the frame of reference rotate with the hollow sphere, then for small values of ω' the centrifugal force will first decrease and will reach a minimum when ω'/ω reaches the value of the drag coefficient. From then on it grows again until ω'/ω equals twice the drag coefficient;[41] then it again declines to the original value it had at ω' = 0. With increasing ω' it increases again until it reaches, for large ω', an amount which differs only slightly from that which it would have without the presence of the hollow sphere, (that is, $\omega'^2\sqrt{(x^2 + y^2)}$), in agreement with our assumption that 2kM/a is much less than unity. That the right-hand members of the equations of motion 25) depend not only on the difference ω-ω' seems at first sight to contradict the nature of the theory of relativity. However, we should not forget that in the problem discussed here we are not only dealing with two bodies (the point mass and the hollow sphere), but the fact is that as a result of the boundary conditions $\gamma_{\mu\nu}=0$, the more distant masses have to be taken into account as a third factor in determining the field; those masses being at rest with respect to our initially-chosen coordinate system.

Summary

By means of a concrete example it has been shown that in an Einsteinian gravitational field, caused by distant, rotating masses, forces appear which are analogous to the centrifugal and Coriolis forces. The peculiarities connected with this special case are throughly discussed.